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The Almost Everywhere Convergence of Eigenfunction Expansions of Schrödinger Operator in L_p Classes.

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ABSTRACT

In this paper the eigenfunction expansions of the Schrödinger operator with the potential having singularity at one point are considered. The uniform estimations for the spectral function of the Schrödinger operator in closed domain are obtained. The almost everywhere convergence of the eigenfunction expansions by Riesz means in the classes L_p classes is proven by estimating the maximal operator in L_1 and L_2 and applying the interpolation theorem for the family of linear operators.

Keywords: Schrödinger operator, almost everywhere convergence and eigenfunction expansions.

1. Introduction

Let Ω be an arbitrary bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary. Let fix the point $x_0 \in \Omega$. We assume that the potential function $q \in L_2(\Omega)$ has a following form:

$$q(x) = \frac{a(|x - x_0|)}{|x - x_0|}, \qquad x \in \Omega,$$

where $a \in C^\infty(0, \infty)$ is non-negative function satisfying the condition:

$$t^k \left| \frac{d^k a(t)}{dt^k} \right| \leq Ct^{\tau - 1}, \qquad k = 0, 1, 2, ..., [N/2]$$

for some $\tau > 0$.

We consider Schrödinger operator $H = -\Delta + q$ with domain $C_0^\infty(\Omega)$. We denote an arbitrary nonnegative self-adjoint extension of H with discrete spectrum by $\hat{H}$. Let $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ...$ be the eigenvalues of $\hat{H}$, and let $\{u_n\}_{n=1}^\infty$ be the corresponding complete orthonormal system of eigenfunctions.

For each $\mathrm{Re}(s) \geq 0$, we define the $s - th$ order Riesz mean of the eigenfunction expansion of a function $f \in L_2(\Omega)$ as follows

$$E_\lambda^s f(x) = \sum_{\lambda_n < \lambda} \left(1 - \frac{\lambda_n}{\lambda}\right)^s (f, u_n) u_n(x),$$

where (f, u_n) denotes the Fourier coefficients of the function f: $(f, u_n) = \int_\Omega f(y) u_n(y) dy, n = 1, 2, 3, ...$

The current work is devoted to the investigations connected with the problems of the almost everywhere convergence of the eigenfunction expansions of the Schrödinger operator by Riesz means. The main result of the paper is the following theorem:

Theorem 1. *If $f(x) \in L_p(\Omega), 1 \leq p \leq 2$. Then Riesz means $E_\lambda^s f(x)$ of order $s > N\left(\frac{1}{p} - \frac{1}{2}\right)$ almost everywhere in Ω converges to $f(x)$.*

The similar statement for the eigenfunction expansions established by Alimov (1970b). When we consider multiple Fourier series and integrals the condition for the almost everywhere convergent guranteed if $s > (N-1)\left(\frac{1}{p}-\frac{1}{2}\right)$ for the values of $p : 1 \leq p \leq 2$. As we can see the gap between these cases. In paper Bastis (1983) has been constructed the example of the eigenfunction expansion corresponding to the self adjoint extension of the Laplace operator that the Riesz means of eigenfunction expansions of order $s : s < N\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{3}$ almost everywhere diverges to infinity for certain function from L_p. In this work we are extending the result of the Alimov (1970b) for the case of Schrodinger operator with nonsmooth potential. To prove the statement in the Theorem 1 we estimate the maximal operator $E_*^s f = \sup_{\lambda>0} |E_\lambda^s f|$ in the classes $L_1(\Omega)$ and $L_2(\Omega)$, and apply interpolation to the family of linear operators.

In this work we investigate the convergence almost everywhere of the eigenfunction expansions of the Schrödinger operator on closed domain where $E_\lambda^s f(x)$ converge to $f(x) \in L_p(\Omega)$.

2. Preliminaries

In this section we obtain a suitable representation for the Riesz means of the eigenfunction expansions of the Schrodinger operator. The eigenfunction expansions of the Schrödinger operator with special type potential function are considered in Alimov and Joo (1983a) and Alimov and Joo (1983b).

Let denote by $r = |x - x_0|, x \in \Omega$. Mean value formula for the eigenfunctions of the Schrödinger operator has following form

$$\int_\theta u_n(x_0+r\theta)d\theta = u_n(x_0)\left[2^{\frac{N-2}{2}}\Gamma\left(\frac{N}{2}\right)(r\sqrt{\lambda_n})^{\frac{2-N}{2}}J_{\frac{N-2}{2}}(r\sqrt{\lambda_n}) + \nu(r,\sqrt{\lambda_n})\right] \tag{1}$$

where the function $\nu(r,\sqrt{\lambda_n})$ satisfies the estimation

$$|\nu(r,\sqrt{\lambda_n})| \leq const(\sqrt{\lambda_n})^{-\tau}\omega(r\sqrt{\lambda_n})$$

here $\omega(t) = \min\{1, \left(\frac{1}{t}\right)^{\frac{N-1}{2}}\}, t > 0$.

For arbitrary $R > 0$ we define the following set: $\Omega_R = \{y \in \Omega : dist(y, \partial\Omega) > R\}$ and for any $x, y \in \Omega$ we introduce the following function:

$$\overline{R}_s(x,y,\lambda) = \begin{cases} \dfrac{2^s\Gamma(s+1)}{(2\pi)^{\frac{N}{2}}}(\sqrt{\lambda})^N \dfrac{J_{\frac{N}{2}+s}(r\sqrt{\lambda})}{(r\sqrt{\lambda})^{\frac{N}{2}+s}}, & |x-y| \leq R \\ 0, \quad |x-y| > R, \end{cases}$$

and

$$D_s(x,y,\lambda) = 2^s\Gamma(s+1)\lambda^{\frac{N}{4}-\frac{s}{2}} \sum_{n=0}^{\infty} \frac{u_n(x)u_n(y)}{(\sqrt{\lambda_n})^{\frac{N}{2}-1}} I_s(\lambda,\lambda_n), \tag{2}$$

where $I_s(\lambda,\lambda_n)$ denotes the integral

$$I_s(\lambda,\lambda_n) = \int_R^{\infty} J_{\frac{N}{2}+s}(r\sqrt{\lambda})J_{\frac{N}{2}-1}(r\sqrt{\lambda_n})r^{-s}dr.$$

The following representation is very important for the estimation of the maximal operator in the classes of $L_p(\Omega)$.

Lemma 2. *The Riesz means has the following representation*

$$E_\lambda^s f(x) = \int_{r\leq R} f(x)\overline{R}_s(x,y,\lambda)dx + \int_\Omega f(x)D_s(x,y,\lambda)dx - \int_\Omega f(x)P_s(x,y,\lambda)dx,$$

where

$$P_s(x,y,\lambda) = \frac{2^s\Gamma(s+1)}{(2\pi)^{\frac{N}{2}}} \sum_{n=0}^{\infty} u_n(x)u_n(y) \int_0^R \frac{J_{\frac{N}{2}+s}(r\sqrt{\lambda})}{(r\sqrt{\lambda})^{\frac{N}{2}+s}}(\sqrt{\lambda})^N \nu(r,\sqrt{\lambda_n})r^{N-1}dr.$$

Proof : Applying the Parseval formula for $f \in L_2(\Omega)$ and

$$g_{x_0}(x) = \begin{cases} |x-x_0|^{-\frac{N}{2}-s}J_{\frac{N}{2}+s}(|x-x_0|\sqrt{\lambda}), & |x-x_0| \leq R \\ 0, \quad |x-x_0| > R. \end{cases}$$

where the number R is chosen with the condition that the ball of radius R:

$$B_R(x_0) = \{x : |x - x_0| < R\} \subset \Omega. \tag{3}$$

We obtain

$$\int_\Omega f(x) g_{x_0}(x) dx = \int_{|x-x_0|\le R} f(x)|x - x_0|^{-\frac{N}{2}-s} J_{\frac{N}{2}+s}(|x - x_0|\sqrt{\lambda}) dy = \sum_{n=0}^{\infty} f_n d_n$$

The Fourier coefficients $d_n = \int_\Omega g_{x_0}(x) u_n(x) dx, n = 1, 2, 3, ...$ of the function $g_{x_0}(x)$ can be calculated using the property that it is radial function and applying mean value formula (1) as follows:

$$\begin{aligned} d_n &= C_N u_n(x_0)(\sqrt{\lambda_n})^{1-\frac{N}{2}} \int_0^R J_{\frac{N}{2}+s}(r\sqrt{\lambda}) J_{\frac{N}{2}-1}(r\sqrt{\lambda_n}) r^{-s} dr \\ &+ u_n(x_0) \int_0^R r^{\frac{N}{2}-s-1} J_{\frac{N}{2}+s}(r\sqrt{\lambda}) \nu(r, \sqrt{\lambda_n}) dr \end{aligned}$$

To the first term of d_n after substitution $\int_0^R = \int_0^\infty - \int_R^\infty$ and taking into account the formula

$$\int_0^\infty J_{\frac{N}{2}+s}(r\sqrt{\lambda}) J_{\frac{N}{2}-1}(r\sqrt{\lambda_n}) r^{-s} dr = \frac{2^{-s} \lambda_n^{\frac{N}{4}-\frac{1}{2}}}{\lambda^{\frac{N}{4}-\frac{s}{2}} \Gamma(s+1)} \left(1 - \frac{\lambda_n}{\lambda}\right)^s \delta_n, \tag{4}$$

where δ_n is a Kronecker we have

$$\begin{aligned} d_n &= C' u_n(x_0) \lambda^{\frac{s}{2}-\frac{N}{4}} \left(1 - \frac{\lambda_n}{\lambda}\right)^s \delta_n - C_N u_n(x_0)(\sqrt{\lambda_n})^{1-\frac{N}{2}} I_s(\lambda, \lambda_n) \\ &+ u_n(x_0) \int_0^R r^{\frac{N}{2}-s-1} J_{\frac{N}{2}+s}(r\sqrt{\lambda}) \nu(r, \sqrt{\lambda_n}) dr \end{aligned}$$

where $C' = \frac{C_N 2^{-s}}{\Gamma(s+1)}$, $C_N = 2^{\frac{N}{2}-1}\Gamma(\frac{N}{2})$, then, $C' = 2^{\frac{N}{2}-s-1}\frac{\Gamma(\frac{N}{2})}{\Gamma(s+1)}$. Substitute d_n into the Parseval formula :

$$\begin{aligned}\int_\Omega fgdx &= \int_{r\leq R} f(x)r^{-\frac{N}{2}-s}J_{\frac{N}{2}+s}(r\sqrt{\lambda})dx \\ &= C'\lambda^{\frac{s}{2}-\frac{N}{4}}\sum_{\lambda_n\leq\lambda} c_n u_n(x_0)\left(1-\frac{\lambda_n}{\lambda}\right)^s - C_N\sum_{n=0}^{\infty} c_n u_n(x_0)(\sqrt{\lambda_n})^{1-\frac{N}{2}} I_s(\lambda,\lambda_n) \\ &+ \sum_{n=0}^{\infty} c_n u_n(x_0)\int_0^R r^{\frac{N}{2}-s-1}J_{\frac{N}{2}+s}(r\sqrt{\lambda})\nu(r,\sqrt{\lambda_n})dr.\end{aligned} \tag{5}$$

We obtain the following representation for the Riesz means:

$$\begin{aligned}E_\lambda^s f(x_0) &= \frac{1}{C'\lambda^{\frac{s}{2}-\frac{N}{4}}}\int_{r\leq R} f(x)r^{-\frac{N}{2}-s}J_{\frac{N}{2}+s}(r\sqrt{\lambda})dx \\ &+ \frac{C_N}{C'\lambda^{\frac{s}{2}-\frac{N}{4}}}\sum_{n=0}^{\infty} c_n u_n(x_0)(\sqrt{\lambda_n})^{1-\frac{N}{2}} I_s(\lambda,\lambda_n) \\ &- \frac{1}{C'\lambda^{\frac{s}{2}-\frac{N}{4}}}\sum_{n=0}^{\infty} c_n u_n(x_0)\int_0^R r^{\frac{N}{2}-s-1}J_{\frac{N}{2}+s}(r\sqrt{\lambda})\nu(r,\sqrt{\lambda_n})dr\end{aligned}$$

which completes proof of Lemma 2.

3. Estimation for $P_s(x,y,\lambda)$

First two terms of Riesz means can be estimated by the methods in Alimov (1970a). Let estimate third term.

Lemma 3. *We have*

$$\int_\Omega |P_s(x,y,\lambda)|^2 dx \leq C\lambda^{\frac{N}{2}-\tau}, \qquad \tau > \frac{N}{2}. \tag{6}$$

uniformly for all $x \in \Omega$ and $\lambda > 1$.

Proof: Let expand $P_s(x,y,\lambda)$ into eigenfunction expansions by eigenfunction $\{u_n(x)\}$:

$$P_s(x,y,\lambda) = \sum_{n=0}^{\infty} c_n(y) u_n(x)$$

where

$$c_n(y) = C_N u_n(y) \int_0^R \frac{J_{\frac{N}{2}+s}(r\sqrt{\lambda})}{(r\sqrt{\lambda})^{\frac{N}{2}+s}} (\sqrt{\lambda})^N \nu(r,\sqrt{\lambda_n}) r^{N-1} dr$$

Application of Parseval's formula gives:

$$\begin{aligned}
\int_\Omega |P_s(x,y,\lambda)|^2 dx = \sum_{n=0}^{\infty} |c_n(y)|^2 = C \sum_{n=0}^{\infty} u_n^2(y) \lambda^N \left| \int_0^R \frac{J_{\frac{N}{2}+s}(r\sqrt{\lambda})}{(r\sqrt{\lambda})^{\frac{N}{2}+s}} \nu(r,\sqrt{\lambda_n}) r^{N-1} dr \right|^2 \\
\leq C \sum_{\lambda_n < \lambda} u_n^2(y) \lambda^N \left| \int_0^R \frac{J_{\frac{N}{2}+s}(r\sqrt{\lambda})}{(r\sqrt{\lambda})^{\frac{N}{2}+s}} \nu(r,\sqrt{\lambda_n}) r^{N-1} dr \right|^2 \\
+ C \sum_{\lambda_n > \lambda} u_n^2(y) \lambda^N \left| \int_0^R \frac{J_{\frac{N}{2}+s}(r\sqrt{\lambda})}{(r\sqrt{\lambda})^{\frac{N}{2}+s}} \nu(r,\sqrt{\lambda_n}) r^{N-1} dr \right|^2
\end{aligned} \tag{7}$$

Let denote

$$|I| = \int_0^R \left| \frac{J_{\frac{N}{2}+s}(r\sqrt{\lambda})}{(r\sqrt{\lambda})^{\frac{N}{2}+s}} \nu(r,\sqrt{\lambda_n}) r^{N-1} \right| dr$$

where $|\nu(r,\sqrt{\lambda_n})| \leq C_\nu (\sqrt{\lambda_n})^{-\tau} \omega(r\sqrt{\lambda_n})$, C_ν is a constant:

$$\omega(r\sqrt{\lambda_n}) = \begin{cases} 1, & 0 < r\sqrt{\lambda_n} < 1 \\ \left(\dfrac{1}{r\sqrt{\lambda_n}}\right)^{\frac{N-1}{2}}, & r\sqrt{\lambda_n} > 1 \end{cases}$$

We have the following lemma to estimate $|I|$ in (7):

Lemma 4. *Let $s > \frac{N}{2}$. For any $r > 0$ we have*

$$|I| = \left|\int_0^R \frac{J_{\frac{N}{2}+s}(r\sqrt{\lambda})}{(r\sqrt{\lambda})^{\frac{N}{2}+s}} \nu(r, \sqrt{\lambda_n}) r^{N-1} dr\right| = O(1) \begin{cases} \lambda_n^{-\frac{\tau+N}{2}}, & \lambda_n < \lambda \\ \lambda^{-\frac{\tau+N}{2}}, & \lambda_n \geq \lambda. \end{cases}$$

Proof: Firstly, let estimate $|I|$ for the case $\lambda_n < \lambda; r < \frac{1}{\sqrt{\lambda}}$:

$$\int_0^{\frac{1}{\sqrt{\lambda}}} \left|\frac{J_{\frac{N}{2}+s}(r\sqrt{\lambda})}{(r\sqrt{\lambda})^{\frac{N}{2}+s}} \nu(r, \sqrt{\lambda_n}) r^{N-1}\right| dr \leq \int_0^{\frac{1}{\sqrt{\lambda}}} \frac{(r\sqrt{\lambda})^{\frac{N}{2}+s}}{(r\sqrt{\lambda})^{\frac{N}{2}+s}} (\sqrt{\lambda_n})^{-\tau} r^{N-1} dr$$
$$= C(\sqrt{\lambda_n})^{-\tau} \int_0^{\frac{1}{\sqrt{\lambda}}} r^{N-1} dr = \frac{C}{N}\left(\frac{1}{\sqrt{\lambda_n}}\right)^{\tau}\left(\frac{1}{\sqrt{\lambda}}\right)^{N} \leq \frac{C}{N}\lambda_n^{-\frac{\tau+N}{2}}.$$

Secondly, let estimate $|I|$ for the case $\lambda_n < \lambda; \lambda^{-\frac{1}{2}} \leq r < \lambda_n^{-\frac{1}{2}}$:

$$\int_{\frac{1}{\sqrt{\lambda}}}^{\frac{1}{\sqrt{\lambda_n}}} \left|\frac{J_{\frac{N}{2}+s}(r\sqrt{\lambda})}{(r\sqrt{\lambda})^{\frac{N}{2}+s}} \nu(r, \sqrt{\lambda_n}) r^{N-1}\right| dr \leq C \int_{\frac{1}{\sqrt{\lambda}}}^{\frac{1}{\sqrt{\lambda_n}}} \frac{(r\sqrt{\lambda})^{-\frac{1}{2}}}{(r\sqrt{\lambda})^{\frac{N}{2}+s}} (\sqrt{\lambda_n})^{-\tau} r^{N-1} dr$$
$$= C(\sqrt{\lambda_n})^{-\tau}\left(\frac{1}{\sqrt{\lambda}}\right)^{\frac{N+1}{2}+s} \int_{\frac{1}{\sqrt{\lambda}}}^{\frac{1}{\sqrt{\lambda_n}}} r^{N-1-\frac{N}{2}-s-\frac{1}{2}} dr = C\left(\frac{1}{\sqrt{\lambda_n}}\right)^{\tau}\left(\frac{1}{\sqrt{\lambda}}\right)^{\frac{N+1}{2}+s} \int_{\frac{1}{\sqrt{\lambda}}}^{\frac{1}{\sqrt{\lambda_n}}} r^{\frac{N}{2}-s-\frac{3}{2}} dr$$
$$\leq \frac{C}{\left(\frac{N-1}{2}-s\right)}\left(\frac{1}{\sqrt{\lambda_n}}\right)^{\frac{N-1}{2}-s+\tau}\left(\frac{1}{\sqrt{\lambda}}\right)^{\frac{N+1}{2}+s} - \frac{C}{\left(\frac{N-1}{2}-s\right)}\left(\frac{1}{\sqrt{\lambda_n}}\right)^{\tau}\left(\frac{1}{\sqrt{\lambda}}\right)^{N}$$
$$\leq \frac{C}{\left(\frac{N-1}{2}-s\right)}\lambda_n^{-\frac{\tau+N}{2}}.$$

Thirdly, let estimate $|I|$ for the case $\lambda_n < \lambda; r > \lambda_n^{-\frac{1}{2}}$:

$$\int_{\frac{1}{\sqrt{\lambda_n}}}^{\infty} \left| \frac{J_{\frac{N}{2}+s}(r\sqrt{\lambda})}{(r\sqrt{\lambda})^{\frac{N}{2}+s}} \nu(r, \sqrt{\lambda_n}) r^{N-1} \right| dr \leq C \int_{\frac{1}{\sqrt{\lambda_n}}}^{\infty} \frac{(r\sqrt{\lambda})^{-\frac{1}{2}}}{(r\sqrt{\lambda})^{\frac{N}{2}+s}} (\sqrt{\lambda_n})^{-\tau} \left(\frac{1}{r\sqrt{\lambda_n}}\right)^{\frac{N-1}{2}} r^{N-1} dr$$
$$= C \left(\frac{1}{\sqrt{\lambda_n}}\right)^{\frac{N-1}{2}+\tau} \left(\frac{1}{\sqrt{\lambda}}\right)^{\frac{N+1}{2}+s} \int_{\frac{1}{\sqrt{\lambda_n}}}^{\infty} \frac{r^{-\frac{1}{2}}}{r^{\frac{N}{2}+s}} \left(\frac{1}{r}\right)^{\frac{N-1}{2}} r^{N-1} dr$$
$$= C \left(\frac{1}{\sqrt{\lambda_n}}\right)^{\frac{N-1}{2}+\tau} \left(\frac{1}{\sqrt{\lambda}}\right)^{\frac{N+1}{2}+s} \int_{\frac{1}{\sqrt{\lambda_n}}}^{\infty} r^{-1-s} dr = C \left(\frac{1}{\sqrt{\lambda_n}}\right)^{\frac{N-1}{2}+\tau} \left(\frac{1}{\sqrt{\lambda}}\right)^{\frac{N+1}{2}+s} \left[\frac{r^{-s}}{-s}\right]_{\frac{1}{\sqrt{\lambda_n}}}^{\infty}$$
$$= \frac{C}{s} \left(\frac{1}{\sqrt{\lambda_n}}\right)^{\frac{N-1}{2}+\tau-s} \left(\frac{1}{\sqrt{\lambda}}\right)^{\frac{N+1}{2}+s} \leq \frac{C}{s} \lambda_n^{-\frac{\tau+N}{2}}.$$

Fourthly, let estimate $|I|$ for the case $\lambda_n > \lambda; r < \frac{1}{\sqrt{\lambda_n}}$:

$$\int_0^{\frac{1}{\sqrt{\lambda_n}}} \left| \frac{J_{\frac{N}{2}+s}(r\sqrt{\lambda})}{(r\sqrt{\lambda})^{\frac{N}{2}+s}} \nu(r, \sqrt{\lambda_n}) r^{N-1} \right| dr \leq C \int_0^{\frac{1}{\sqrt{\lambda_n}}} \frac{(r\sqrt{\lambda})^{\frac{N}{2}+s}}{(r\sqrt{\lambda})^{\frac{N}{2}+s}} (\sqrt{\lambda_n})^{-\tau} r^{N-1} dr$$
$$= C(\sqrt{\lambda_n})^{-\tau} \int_0^{\frac{1}{\sqrt{\lambda_n}}} r^{N-1} dr = \frac{C}{N} \left(\frac{1}{\sqrt{\lambda_n}}\right)^{N+\tau} \leq \frac{C}{N} \lambda^{-\frac{\tau+N}{2}}.$$

Fifthly, let estimate $|I|$ for the case $\lambda_n > \lambda; \lambda_n^{-\frac{1}{2}} \leq r < \lambda^{-\frac{1}{2}}$:

$$C \int_{\frac{1}{\sqrt{\lambda_n}}}^{\frac{1}{\sqrt{\lambda}}} \left| \frac{J_{\frac{N}{2}+s}(r\sqrt{\lambda})}{(r\sqrt{\lambda})^{\frac{N}{2}+s}} \nu(r, \sqrt{\lambda_n}) r^{N-1} \right| dr \leq C \int_{\frac{1}{\sqrt{\lambda_n}}}^{\frac{1}{\sqrt{\lambda}}} \frac{(r\sqrt{\lambda})^{\frac{N}{2}+s}}{(r\sqrt{\lambda})^{\frac{N}{2}+s}} (\sqrt{\lambda_n})^{-\tau} \left(\frac{1}{r\sqrt{\lambda_n}}\right)^{\frac{N-1}{2}} r^{N-1} dr$$
$$= C \left(\frac{1}{\sqrt{\lambda_n}}\right)^{\frac{N-1}{2}+\tau} \int_{\frac{1}{\sqrt{\lambda_n}}}^{\frac{1}{\sqrt{\lambda}}} r^{\frac{N-1}{2}} dr = C \left(\frac{1}{\sqrt{\lambda_n}}\right)^{\frac{N-1}{2}+\tau} \left[\frac{r^{\frac{N+1}{2}}}{\frac{N+1}{2}}\right]_{\frac{1}{\sqrt{\lambda_n}}}^{\frac{1}{\sqrt{\lambda}}}$$
$$\leq \frac{C}{(N+1)} \left(\frac{1}{\sqrt{\lambda_n}}\right)^{\frac{N-1}{2}+\tau} \left[\left(\frac{1}{\sqrt{\lambda}}\right)^{\frac{N+1}{2}} - \left(\frac{1}{\sqrt{\lambda_n}}\right)^{\frac{N+1}{2}}\right]$$
$$= \frac{C}{(N+1)} \left(\frac{1}{\sqrt{\lambda_n}}\right)^{\frac{N-1}{2}+\tau} \left(\frac{1}{\sqrt{\lambda}}\right)^{\frac{N+1}{2}} - \frac{C}{(N+1)} \left(\frac{1}{\sqrt{\lambda_n}}\right)^{N+\tau} \leq \frac{C}{N+1} \lambda^{-\frac{\tau+N}{2}}.$$

Lastly, let estimate $|I|$ for the case $\lambda_n > \lambda; r > \lambda^{-\frac{1}{2}}$:

$$C\int_{\frac{1}{\sqrt{\lambda}}}^{\infty}\left|\frac{J_{\frac{N}{2}+s}(r\sqrt{\lambda})}{(r\sqrt{\lambda})^{\frac{N}{2}+s}}\nu(r,\sqrt{\lambda_n})r^{N-1}\right|dr \leq C\int_{\frac{1}{\sqrt{\lambda}}}^{\infty}\frac{(r\sqrt{\lambda})^{-\frac{1}{2}}}{(r\sqrt{\lambda})^{\frac{N}{2}+s}}(\sqrt{\lambda_n})^{-\tau}\left(\frac{1}{r\sqrt{\lambda_n}}\right)^{\frac{N-1}{2}}r^{N-1}dr$$
$$= C\left(\frac{1}{\sqrt{\lambda_n}}\right)^{\frac{N-1}{2}+\tau}\left(\frac{1}{\sqrt{\lambda}}\right)^{\frac{N+1}{2}+s}\int_{\frac{1}{\sqrt{\lambda}}}^{\infty}\frac{r^{-\frac{1}{2}}}{r^{\frac{N}{2}+s}}\left(\frac{1}{r}\right)^{\frac{N-1}{2}}r^{N-1}dr$$
$$= C\left(\frac{1}{\sqrt{\lambda_n}}\right)^{\frac{N-1}{2}+\tau}\left(\frac{1}{\sqrt{\lambda}}\right)^{\frac{N+1}{2}+s}\int_{\frac{1}{\sqrt{\lambda}}}^{\infty}r^{-1-s}dr = C\left(\frac{1}{\sqrt{\lambda_n}}\right)^{\frac{N-1}{2}+\tau}\left(\frac{1}{\sqrt{\lambda}}\right)^{\frac{N+1}{2}+s}\left[\frac{r^{-s}}{-s}\right]_{\frac{1}{\sqrt{\lambda}}}^{\infty}$$
$$= \frac{C}{s}\left(\frac{1}{\sqrt{\lambda_n}}\right)^{\frac{N-1}{2}+\tau}\left(\frac{1}{\sqrt{\lambda}}\right)^{\frac{N+1}{2}} \leq \frac{C}{s}\lambda^{-\frac{\tau+N}{2}}.$$

Finally, we obtain

$$\begin{aligned}\int_{\Omega}|P_s(x,y,\lambda)|^2dx &= \sum_{n=0}^{\infty}|p_n(y)|^2 \\ &= C\sum_{n=0}^{\infty}u_n^2(y)\lambda^N\left|\int_0^R\frac{J_{\frac{N}{2}+s}(r\sqrt{\lambda})}{(r\sqrt{\lambda})^{\frac{N}{2}+s}}\nu(r,\sqrt{\lambda_n})r^{N-1}dr\right|^2 \quad (8) \\ &\leq C_1\sum_{\lambda_n<\lambda}u_n^2(y)\lambda_n^{-\tau} + C_2\sum_{\lambda_n>\lambda}u_n^2(y)\lambda^{-\tau}.\end{aligned}$$

where $C_1 = \left[\frac{C}{N} + \frac{C}{(\frac{N-1}{2}-s)} + \frac{C}{s}\right]^2$ *and* $C_2 = \left[\frac{C}{N} + \frac{C}{N+1} + \frac{C}{s}\right]^2$.

We refer the following formula from Alimov and Joo (1983b)

$$\sum_{|\sqrt{\lambda_n}-\lambda|\leq 1}u_n^2(x) = O(1)\lambda^{N-1}, \lambda > 1, \qquad \forall x\in\Omega.$$

firstly, we have to estimate the first term of (8) as follows

$$C_1 \sum_{\lambda_n<\lambda} u_n^2(y)\lambda_n^{-\tau} \leq C_1 \sum_{m=1}^{[\sqrt{\lambda}]} m^{-2\tau} \sum_{m<\sqrt{\lambda_n}<m+1} u_n^2(y)$$

$$\leq C_1 \sum_{m=1}^{[\sqrt{\lambda}]} m^{-2\tau} m^{N-1} = C_1 \int_1^{[\sqrt{\lambda}]} t^{-2\tau+N-1} dt$$

$$= C_1 \left[\frac{t^{-2\tau+N}}{-2\tau+N}\right]_1^{[\sqrt{\lambda}]} = \frac{C_1}{N-2\tau}[\lambda^{-\tau+\frac{N}{2}} - 1] \leq C_3\lambda^{\frac{N}{2}-\tau}.$$

where $C_3 = \frac{C_1}{N-2\tau}$.

Next we estimate the second term of equation (8),

$$C_2 \sum_{\lambda_n>\lambda} u_n^2(y)\lambda^{-\tau} \leq C_2 \sum_{m=[\sqrt{\lambda}]}^{\infty} m^{-2\tau} \sum_{m<\sqrt{\lambda_n}<m+1} u_n^2(y) \leq C_2 \sum_{m=[\sqrt{\lambda}]}^{\infty} m^{-2\tau} m^{N-1}$$

$$\leq C_2 \int_{\sqrt{\lambda}}^{\infty} t^{-2\tau+N-1} dt = C_2 \left[\frac{t^{N-2\tau}}{N-2\tau}\right]_{\sqrt{\lambda}}^{\infty} = C_4\lambda^{\frac{N}{2}-\tau}$$

where $C_4 = \frac{C_2}{N-2\tau}$.

Then,

$$\int_{\overline{\Omega}} |P_s(x,y,\lambda)|^2 dx \leq C\lambda^{\frac{N}{2}-\tau}, \qquad \tau > \frac{N}{2}, s > \frac{N}{2},$$

where $C = C_3 + C_4$. This is completing proof.

4. Estimation in $L_p, p > 1$

Lemma 5. *Let*

$$P_s^*(x,y) = \sup_{\sqrt{\lambda}} |P_s(x,y,\sqrt{\lambda})|.$$

If $2\alpha - N = \epsilon > 0, s = \alpha + i\delta,$ *and* $y \in \Omega$ *we have the relations as follows*

$$\int_{\Omega} [P_s^*(x,y)]^2 dx \leq C.$$

Proof. We know

$$|P_s(x,y,\lambda)|^2 = 2\left|\int_0^{\mu} P_s(x,y,t)\frac{\partial}{\partial t}(P_s(x,y,t))dt\right| \leq \int_0^{\mu} |P_s(x,y,t)|^2 + \left|\frac{\partial}{\partial t}(P_s(x,y,t))\right|^2 dt$$

Taking supremum, we have

$$|P_s^*(x,y,)|^2 \leq \int_0^{\mu} |P_s(x,y,t)|^2 + \left|\frac{\partial}{\partial t}(P_s(x,y,t))\right|^2 dt$$

We integrate both sides with respect to x, we obtain

$$\int_{\Omega} |P_s^*(x,y,)|^2 dx \leq \int_{\Omega} \left[\int_0^{\mu} |P_s(x,y,t)|^2 + \left|\frac{\partial}{\partial t}(P_s(x,y,t))\right|^2 dt\right] dx$$
$$= \int_0^{\mu} \left[\int_{\Omega} |P_s(x,y,t)|^2 + \left|\frac{\partial}{\partial t}(P_s(x,y,t))\right|^2 dx\right] dt$$

We have the following estimate if we choose $\tau > \frac{N}{2} + 1$, then we obtain

$$\int_{\Omega} |P_s^*(x,y,\lambda)|^2 dx \leq \int_1^{\infty} \left[\int_{\Omega} |P_s(x,y,t)|^2 + \left|\frac{\partial}{\partial t}(P_s(x,y,t))\right|^2 dx\right] dt = \int_1^{\infty} t^{\frac{N}{2}-\tau} dt = C.$$

Lemma 5 is proved.

5. Estimation in L_1

As we established in the beginning for the Riesz means we have: $E_\lambda^s f(x) = \sigma_\lambda^s f(x) + \xi_\lambda^s f(x) + \gamma_\lambda^s f(x)$.

From Alimov (1970a)'s result we conclude that:

$$||\xi_*^{s+i\delta} f||_{L_2(\Omega)} \leq \frac{C_1}{2s-N}||f||_{L_1(\Omega)}, \qquad \forall f \in L_1(\Omega),$$

$$||\sigma_*^{s+i\delta} f||_{L_p(\Omega)} \leq \frac{C_2}{p-1}||f||_{L_p(\Omega)}, \qquad 1 \leq p \leq 2.$$

Collecting all estimations we obtain

$$||E_*^{s+i\delta} f||_{L_p(\Omega)} \leq C\left(\frac{1}{p-1} + \frac{1}{\sqrt{2s-N}} + \frac{1}{\sqrt{2\tau-N-2}}\right)||f||_{L_p(\Omega)}. \qquad p>1, \qquad (p \to 1). \tag{9}$$

Lemma 6. *The functions*

$$W_s(x,f) = \left[\int_0^\infty |E_\lambda^s f(x) - E_\lambda^{s-1} f(x)|^2 \frac{d\sqrt{\lambda}}{\sqrt{\lambda}}\right]^{\frac{1}{2}}, \qquad Q_s(x,f) = \sup_{\sqrt{\lambda}}\left[\frac{1}{\sqrt{\lambda}}\int_0^{\sqrt{\lambda}} |E_\lambda^{s-1} f(x,t)|^2 dt\right]^{\frac{1}{2}} \tag{10}$$

$Re(s) = \alpha > \frac{1}{2}$.

$$\int_\Omega |W_s(x,f)|^2 dx \leq C\int_\Omega |f(x)|^2 dx, \qquad \int_\Omega |Q_s(x,f)|^2 dx \leq C\int_\Omega |f(x)|^2 dx. \tag{11}$$

Proof. To prove (11) we note that

$$\left[\frac{1}{\sqrt{\lambda}}\int_0^{\sqrt{\mu}} |E_\lambda^{s-1} f(x,t)|^2 dt\right]^{\frac{1}{2}} \leq \sum_{k=0}^{m}\left[\frac{1}{\sqrt{\lambda}}\int_0^{\sqrt{\mu}} |E_\lambda^{s+k} f(x,t) - E_\lambda^{s+k-1} f(x,t)|^2 dt\right]^{\frac{1}{2}}$$
$$+\left[\frac{1}{\sqrt{\lambda}}\int_0^{\sqrt{\mu}} |E_\lambda^{s+m} f(x,t)|^2 dt\right]^{\frac{1}{2}} \leq \sum_{k=0}^{m} W_{s+k}(x;f) + Q_{s+m+1}(x,f). \tag{12}$$

From (10) and (12) the estimate

$$Q_s(x,f) \leq \sum_{k=0}^{m} W_{s+k}(x;f) + E^*_{s+m}(x,f).$$

Choose m such that $m+\alpha > \frac{N}{2}$, it follows from (9) and (11) we obtain (10). From Corollary, for any $s=\alpha+i\delta$,Re$(s)>0$,

$$\int_\Omega |E_\lambda^{s,*}(x,f)|^2 dx \leq C\int_\Omega |f(x)|^2 dx. \tag{13}$$

This follows from the obvious relation

$$E_s(\sqrt{\lambda}) = \sqrt{\lambda}^{-2s}\frac{\Gamma(s+1)}{\left[\Gamma(\frac{s+1}{2})\right]^2}\int_0^{\mu} E_{\frac{s-1}{2}}(t)t^s(\mu^2-t^2)^{\frac{s-1}{2}} dt,$$

$$|E_s(x,s,\sqrt{\lambda})| \leq C\left[\frac{1}{\sqrt{\lambda}}\int_0^{\mu} |E_{\frac{s-1}{2}}(t)|^2 dt\right]^{\frac{1}{2}} \leq CQ_{\frac{s+1}{2}}(x,f). \tag{14}$$

From (14) and (11) implies (13).

6. The analytic family of linear operators and L_p-spaces

We say that a function $\varphi(\tau), \tau \in \mathbb{R}^1$, has admissible growth if there exist constants $a < \pi$ and $b > 0$ such that

$$|\varphi(z)| \leq \exp(b \exp a|\tau|). \tag{15}$$

Let A_z be a family of operators defined for simple functions (i.e. functions which are finite linear combinations of characteristic functions of measurable subsets of Ω). We term the family A_z admissible if for any two simple functions f and g the function

$$\varphi(z) = \int_\Omega f(x) A_z g(x) dx$$

is analytic in the strip $0 \leq Rez \leq 1$ and has admissible growth in Im z, uniformly in Rez (this means that we have an estimate in Imz which is analogous to (15), with constants a and b independent of Rez).

Theorem 7. *Let A_z be an admissible family of linear operators such that*

$$||A_{i\tau} f||_{L_{p_0}(\Omega)} \leq M_0(\tau) ||f||_{L_{p_0}(\Omega)}, \qquad 1 \leq p_0 \leq \infty,$$

$$||A_{1+i\tau} f||_{L_{p_1}(\Omega)} \leq M_1(\tau) ||f||_{L_{p_1}(\Omega)}, \qquad 1 \leq p_1 \leq \infty,$$

for all simple functions f and with $M_j(\tau)$ independent of τ and of admissible growth. Then there exist for each t, $0 \leq t \leq 1$, a constant M_t such that for every simple function f holds

$$||A_t f||_{L_{p_t}(\Omega)} \leq M_t ||f||_{L_{p_t}(\Omega)}, \qquad \frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}.$$

We apply the interpolation theorem on an analytic family of linear operators in L_p space. Let $\mu(x)$ be a measurable function on Ω such as $0 \leq \mu(x) \leq \mu_0 <$

∞ and $\alpha(z) = \left(\frac{N}{2} + \epsilon\right) z$, $0 \leq Rez \leq 1$. We define an analytic family of linear operators:

$$A_z f(x) = E^{\alpha(z)}_{\mu(x)} f(x), \qquad 0 \leq Rez \leq 1.$$

We have

$$||A_{iy} f(x)||_{L_2(\Omega)} \leq ||E^{\alpha(iy)}_{*} f(x)||_{L_2(\Omega)} \leq Be^{\pi \frac{|y|}{2}} ||f||_{L_2(\Omega)}.$$

Secondly on the line $z = 1 + iy$, we have

$$||A_{1+iy} f(x)||_{L_{p_0}(\Omega)} \leq ||E^{\alpha(1+iy)}_{*} f(x)||_{L_{p_0(\Omega)}} \leq De^{\pi \frac{|y|}{2}} ||f||_{L_{p_0}(\Omega)}.$$

Therefore by the interpolation we get

$$||E^{\alpha t}_{*} f(x)||_{L_\Omega} \leq M_t ||f||_{L_p(\Omega)}. \tag{16}$$

And if note that

$$\frac{1}{p} = \frac{1-t}{2} + \frac{t}{p_0}, \qquad \frac{1}{p} - \frac{1}{2} = t\left(\frac{1}{p_0} - \frac{1}{2}\right),$$

then,

$$t = \frac{\frac{1}{p} - \frac{1}{2}}{\frac{1}{p_0} - \frac{1}{2}} > \frac{\frac{1}{p} - \frac{1}{2}}{\frac{1}{2}} = 2\left(\frac{1}{p} - \frac{1}{2}\right),$$

then,

$$\alpha(t) = \left(\frac{N}{2} + \epsilon\right) t > \left(\frac{N}{2} + \epsilon\right) 2\left(\frac{1}{p} - \frac{1}{2}\right) > N\left(\frac{1}{p} - \frac{1}{2}\right), \qquad p_0 > 1.$$

Let us denote by $\Lambda f(x)$ the fluctuation of $E_n^s f(x)$:

$$\Lambda f(x) = |\lim_{\lambda\to\infty} \sup E_\lambda^s f(x) - \lim_{\lambda\to\infty} \inf E_\lambda^s f(x)|.$$

It is obvious, that $\Lambda f(x) \leq E_*^s f(x)$.

From density of $C^\infty \in L_p, p \geq 1$, we have that for any $\epsilon > 0$ the function $f \in L_p, p \geq 1$ can be represented as the sum of two functions: $f(x) = f_1(x) + f_2(x)$, where $f_x \in C^\infty$, and $||f_2||_{L_2} \leq \epsilon$. Then we have

$$\Lambda f(x) = |\lim_{\lambda\to\infty} \sup E_\lambda^s f_2(x) - \lim_{\lambda\to\infty} \inf E_\lambda^s f(x)| \leq ||f_2||_{L_p} \leq \epsilon.$$

Therefore almost everywhere $\Lambda f(x) = 0$. So almost everywhere on Ω for Riesz means of order $s > N\left(\frac{1}{p} - \frac{1}{2}\right)$, $1 \leq p \leq 2$, we have $\lim_{\lambda\to\infty} E_\lambda^s f(x) = 0$.

7. Conclusion

The almost everywhere convergence of the eigenfunction expansions of the Schrödinger operator by Riesz means of order $s > N(1/p - 1/2)$ in the classes of L_p, $1 \leq p \leq 2$ is established by estimating the maximal operators in the classes L_1 and L_2 and application of the interpolation Theorem for the family of linear operators. This result is extending the similar result for the eigenfunction expansions of the Laplace operator obtained in the work Alimov (1970b).

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